# Multi Variable Optimization

## Simple Two Variable Optimization

First order conditions: Two Variables:
A necessary condition for a differentiable function f(x, y) to have a maximum or minimum at an interior point $(x_0, y_0)$ of its domain is that $(x_0, y_0)$ is a stationary point of f — that is:

$$f_1'(x_0, y_0) = 0,$$
$$f_2'(x_0, y_0) = 0.$$

## Maxima and Minima

Let f be a function of n variables $x_1, \ldots, x_n$ defined over a set S in $\mathbb{R}^n$. Suppose that $c = (c_1, \ldots, c_n)$ belongs to S and gives a value to f that is larger than or equal to the values attained by f at all other points $x = (x_1, \ldots, x_n)$ of S. Thus in symbols,

$$f(x) \leq f(c) \quad \forall\, x \in S.$$

Then c is called a (global) maximum point for f in S and f(c) is called the max. value. By reversing the inequality sign, we obtain minimum pt and min value. They are collectively known as extreme points and extreme values.

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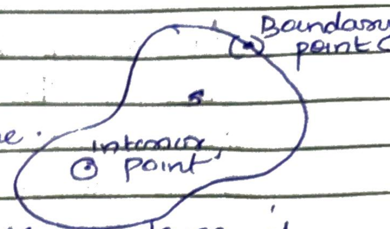
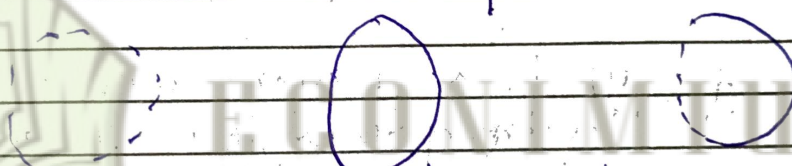
# Topology in the Plane.




We start with sets in the plane. A point $(a,b)$ is called an interior point of a set $S$ in the plane if there exists a circle centered @ $(a,b)$ s all points inside the circle lie in $S$. A set is called open if it consists of only interior points. The point $(a,b)$ is called boundary point of a set $S$ if every circle centered at $(a,b)$ contains points inside $S$ as well as outside $S$. However if $S$ contains all its boundary points then $S$ is called closed.



open closed Neither open nor closed.



A set is bounded if the whole set is contained within a sufficiently large circle.
~~The sets in eg. 12.4.4 and the Fig 2.4.8~~
The set in the plane that is closed and bounded is called compact.

## Extreme Value Theorem.

If $f$ is a continuous function over a closed bounded set $S$ in $\mathbb{R}^n$, then there exist both a maximum $c = (c_1, \ldots, c_n)$ and a minimum point $d = (d_1, \ldots, d_n)$ in $S$ – that is, there exist $c$ and $d$ in $S$ such that

$$f(d) \leq f(x) \leq f(c) \quad \forall x \text{ in } S.$$

**Necessary First-order conditions**

Let $f$ be defined in a set $S$ in $\mathbb{R}^n$ and let $c = (c_1, \dots, c_n)$ be an interior point in $S$ at which $f$ is differentiable. A necessary condition for $c$ to be a maximum or minimum point for $f$ is that $c$ is a stationary point for $f$ — that is,

$$f_i'(c) = 0 \quad (i = 1 \dots n)$$

**Second Derivative Test for Functions of Two Variables.**

Let $f(x, y)$ be a function with continuous partial derivatives of the first and second order in a domain $S$ and let $(x_0, y_0)$ be an interior point of $S$ that is stationary point for $f$. Write

$$A = f_{11}''(x_0, y_0) \quad B = f_{12}''(x_0, y_0) \quad C = f_{22}''(x_0, y_0)$$

Now

(a) If $A < 0$ and $AC - B^2 > 0$, then $(x_0, y_0)$ is a local max point

(b) If $A > 0$ and $AC - B^2 > 0$, then $(x_0, y_0)$ is a local min point

(c) If $AC - B^2 < 0$ then $(x_0, y_0)$ is a saddle point

(d) If $AC - B^2 = 0$, then $(x_0, y_0)$ could be a local max, a local min or a saddle point.

## Convex sets.

A set of points $S$ in the plane is called convex if each pair of points in $S$ can be joined by a line segment lying entirely within $S$.

A set $S$ in $\mathbb{R}^n$ is convex if

$$x \in S,\ y \in S \text{ and } \lambda \in [0,1] \Rightarrow (1-\lambda)x + \lambda y \in S$$

Note :–

If $S$ and $T$ are two convex sets in $\mathbb{R}^n$, then their intersection $S \cap T$ is also convex.

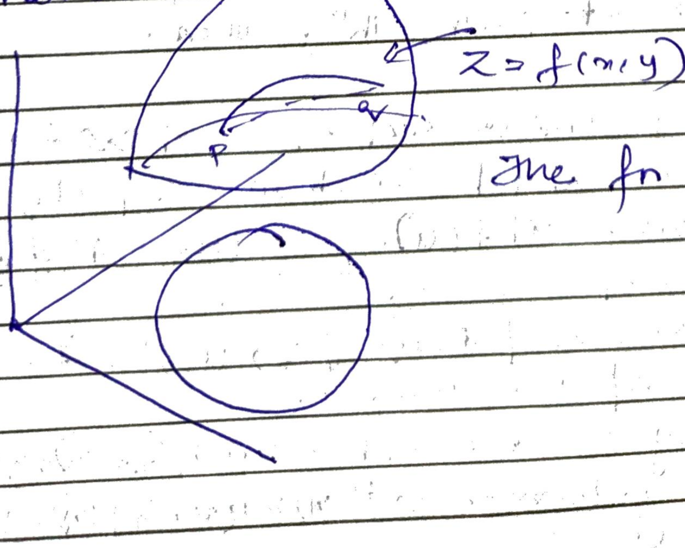
More generally:

$$S_1, \ldots, S_m \text{ convex in } \mathbb{R}^n \Rightarrow S_1 \cap \cdots \cap S_m \text{ convex}$$



## Concave and Convex Functions

The function $f(x,y)$ is concave (convex) if its domain is convex and the line segment joining any two points on the graph is never above (below) the graph




The fn $f(x,y)$ is concave.

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Definition of Concave function

A function $f(x) = f(x_1, \ldots x_n)$ defined on a convex set S is concave in S if

$$f((1-\lambda)x^0 + \lambda x) \geq (1-\lambda)f(x^0) + \lambda f(x)$$

$\forall\ x^0, x \in S$, and all $\lambda \in (0,1)$

Jensen's inequality (Continous Version).

Let $x(t)$ and $\lambda(t)$ be continous functions in the interval $[a,b]$, with $\lambda(t) \geq 0$ and $\int_a^b \lambda(t)\,dt = 1$. If f is a concave function defined on the range of $x(t)$, then

$$f\left(\int_a^b \lambda(t)\, x(t)\, dt\right) \geq \int_a^b \lambda(t)\, f(x(t))\, dt$$

Useful Results.

Let f and g be functions defined over a convex set S in $\mathbb{R}^n$. Then:

f and g concave and $a \geq 0$, $b \geq 0 \Rightarrow af + bg$ concave

f and g convex and $a \geq 0, b \geq 0 \Rightarrow af + bg$ convex

$f(x)$ concave and $F(u)$ concave and increasing $\Big\} \Rightarrow U(x) = F(f(x))$ concave

$f(x)$ convex and $F(u)$ convex and increasing $\Big\} \Rightarrow U(x) = F(f(x))$ convex

f and g concave $\Rightarrow h(x) = \min\{f(x), g(x)\}$ is concave

f and g convex $\Rightarrow H(x) = \max\{f(x), g(x)\}$ is convex

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Theorem

Suppose that $f(x)$ has continuous partial derivatives in a convex set $S$ in $\mathbb{R}^n$, and let $x^0$ be an interior point in $S$. Now:

(a) If $f$ is concave, then $x^0$ is a (global) max point of $f$ in $S$ iff $x^0$ is a stationary point of $f$.

(B) If $f$ is convex, then $x^0$ is a (global) minimum point of $f$ in $S$ iff $x^0$ is a stationary point of $f$

Second Derivative Tests for Concavity / Convexity

The Two Variable Case:

Let $z = f(x, y)$ be a function with continuous partial derivatives of the first and second order, defined over an open convex set $S$ in the plane. Then:

(a) $f$ is concave $\iff f''_{11} \leq 0, f''_{22} \leq 0$ & $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$

(b) $f$ is convex $\Leftrightarrow f''_{11} \geq 0, f''_{22} \geq 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$

(c) $f''_{11} < 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0 \Rightarrow f$ is strictly concave

(d) $f''_{11} > 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0 \Rightarrow f$ is strictly convex

where all inequalities should hold through $S$.

The n-variable case.

Suppose that $z = f(x) = f(x_1, \ldots x_n)$ is a c function in a domain $S$ in $\mathbb{R}^n$.

$$H(x) = [f''_{ij}(x)]_{n \times n}$$

is called the Hessian / Hessian matrix of f at $x$.

$$D_k(x) = \begin{vmatrix} f''_{11}(x) & f''_{12}(x) & \cdots & f''_{1k}(x) \\ f''_{21}(x) & f''_{22}(x) & \cdots & f''_{2k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{k1}(x) & f''_{k2}(x) & \cdots & f''_{kk}(x) \end{vmatrix} \quad (k = 1, \ldots, n)$$

are called leading principal minors of $H(x)$



$(-1)^k D_k(x) > 0 \ \forall \ k = 1 \ldots n$ and $\forall \ x \in S$

$\implies f$ is strictly concave in S

$D_k(x) > 0 \ \forall \ k = 1 \ldots n \ \forall \ x \in S$.

$\implies f$ is strictly convex in S

Saddle Point Test

If $D_n(x^0) \neq 0$ and neither the conditions $(-1)^k D_k(x^0) > 0$ or $D_k(x^0) > 0$ is satisfied then a stationary point $x^0$ of $f$ is a saddle point.

Let $z = f(x)$ be a $C^2$ function in an open convex set $S$ in $R^n$. If $H(x)$ denotes the Hessian matrix of $f$, then:

(a) $f$ is concave $\Leftrightarrow$ $H(x)$ is -ve semidefinite $\forall x \in S$
(b) $f$ is convex $\Leftrightarrow$ $H(x)$ is +ve semidefinite $\forall x \in S$.
(c) $f$ is strictly concave if $H(x)$ is negative def $\forall x \in S$
(d) $f$ is strictly convex if $H(x)$ is +ve def $\forall x \in S$

## Quasi-Concave and Quasi-Convex functions.

Let $f(x)$ be a function defined over a convex set $S$ in $R^n$. For each real number $a$, define the set $P_a$ by

$$P_a = \{x \in S : f(x) \geq a\}$$

Then $P_a$ is a subset of $S$ and is called an upper level set for $f$.

The function $f$, defined over a convex set $S \subset R^n$ is quasi-concave if the upper level set $P_a = \{x \in S : f(x) \geq a\}$ is convex for each number $a$.

$f$ is quasi-convex if $-f$ is quasi-concave.
So $f$ is quasi-convex iff the lower level set $P^a = \{x : f(x) \leq a\}$ is convex for each number $a$.

Result:-
If $f(x)$ is concave, then $f(x)$ is quasi-concave
If $f(x)$ is convex, then $f(x)$ is quasi-convex.

Note :-

> A sum of quasi-concave (quasi convex) functions is not necessarily quasi-concave (quasi convex)

> If $f(x)$ is quasi concave (quasi convex and $F$ is strictly increasing then $F(f(x))$ is quasi-concave (quasi-convex)

> If $f(x)$ is quasi-concave (quasi-convex) and if $F$ is strictly decreasing, then $F(f(x))$ is quasi-convex (quasi-concave)

Constrained Optimization.

Consider the problem of maximising (or minimising) a function $f(x,y)$ when $x$ and $y$ are restricted to satisfy an equation $g(x,y) = c$. In case we want to maximise $f(x,y)$, the problem is

$$\max f(x,y) \text{ subject to } g(x,y) = c.$$

Geometrically, the graph of $f$ is a surface like a bowl, whereas the equation $g(x,y) = c$ represents a curve in the $xy$-plane. The condition for optimization (geometrically) is that the slope of the tangent to the curve $g(x,y) = c$ at $(x,y)$ is equal to the slope of the tangent to the level curve of $f$ at that point.

ie
$$\frac{f_1'(x,y)}{f_2'(x,y)} = \frac{g_1'(x,y)}{g_2'(x,y)}$$

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# The Lagrange Multiplier Method

To find the solutions of the problem
$$\max(\min)\ f(x,y) \text{ subject to } g(x,y) = c$$
proceed as follows:

1. Write down the Lagrangean function
$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$$
where $\lambda$ is a constant.

2. Differentiate $\mathcal{L}$ w.r.t $x$ and $y$ and equate the partials to 0.

3. The two equations in 2 together with the constraint yield the following three equations:
$$f_1'(x,y) = \lambda g_1'(x,y)$$
$$f_2'(x,y) = \lambda g_2'(x,y)$$
$$g(x,y) = c$$

4 Solve these three equations for the three unknowns $x$, $y$ and $\lambda$.



Economic Interpretations

Suppose $x^* = x^*(c)$ and $y^* = y^*(c)$ (are differentiable functions of $c$) are the values of $x$ and $y$ that solves this problem, then
$$f^*(c) = f(x^*(c), y^*(c))$$ is called the (optimal) value function for the problem.
When using Lagrangean method, the associated value $\lambda(c)$ of the Lagrange multiplier also depends on $c$. Provided that regularity conditions are

satisfied, we have the remarkable result that

$$\frac{d f^*(c)}{dc} = \lambda(c).$$

Thus, the Lagrange multiplier $\lambda = \lambda(c)$ is the rate at which the optimal value of the objective function changes w.r.t changes in the constraint constant $c$.

In economic applications, $c$ often denotes the available stock of some resource, and $f(x,y)$ denotes utility or profit. Then $\lambda(c)\,dc$, for $dc > 0$ measures approximately the increase in utility or profit that can be obtained from $dc$ units more of the resource. Economists call $\lambda$ a shadow price of the resource.



Lagrange's Theorem

Suppose that $f(x,y)$ and $g(x,y)$ have continuous partial derivatives in a domain $A$ of the $xy$-plane, and that $(x_0, y_0)$ is both an interior point of $A$ and a local extreme point for $f(x,y)$ subject to the constraint $g(x,y) = c$. Suppose further that $g_1'(x_0,y_0)$ and $g_2'(x_0,y_0)$ are not both $0$. Then there exists a unique number $\lambda$ such that the Lagrangean function

$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$$

has a stationary point at $(x_0, y_0)$.

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# Theorem - Global Sufficiency.

Suppose that $f(x,y)$ and $g(x,y)$ in problems of optimization are continuously differentiable function on an open convex set $A$ in $R^2$ and let $(x_0, y_0) \in A$ be an interior stationary point for the Lagrangean function

$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$$

Suppose further that $g(x_0, y_0) = c$. Then

$\mathcal{L}(x,y)$ concave $\Rightarrow (x_0, y_0)$ solves max. problem

$\mathcal{L}(x,y)$ convex $\Rightarrow (x_0, y_0)$ solves min problem

Note that $\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$ is concave if $f(x,y)$ is concave and $\lambda g(x,y)$ is convex, because then $\mathcal{L}(x,y) = f(x,y) + [-\lambda g(x,y)] + \lambda c$ is a sum of concave functions

# The Lagrangean Multiplier - General Case

Generalizing, the corresponding general Lagrangean problem is

$$\max(\min)\ f(x_1, \dots x_n) \text{ subject to } \begin{cases} g_1(x_1, \dots, x_n) = c_1 \\ \text{---------} \\ g_m(x_1, \dots x_n) = c_m \end{cases}$$

The Lagrangean function is

$$\mathcal{L}(x_1, \dots x_n) = f(x_1, \dots x_n) - \sum_{j=1}^{n} \lambda_j (g_j(x_1, \dots x_n) - c_j)$$

The necessary 1st order conditions for an optimum are that the partial derivatives of the Lagrangean w.r.t each $x_i$ vanish so that

$$\frac{\partial L}{\partial x_i} = \frac{\partial f(x_1, \ldots x_n)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x_1, \ldots x_n)}{\partial x_i} = 0$$

$i = 1, 2, \ldots, n$

Together with m equality constraints, these n equations form a total of n+m equations with n+m unknowns $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$.

Envelope Results.

Consider the problem

$$\max_x f(x, r) \text{ subject to } g_j(x, r) = 0$$

$j = 1, \ldots, m$

where $r = (r_1, \ldots, r_k)$ is a vector of parameters. Here r is kept constant during the maximization w.r.t $x = (x_1, \ldots, x_n)$. Note that the parameters may appear in the objective as well as in the constraint function. The maximum value of $f(x, r)$ obtained in the problem will depend on r and we denote it by $f^*(r)$.

Assuming that the max value exists.

$$f^*(r) = \max\{f(x, r) : g_j(x, r) = 0, j = 1, \ldots, m\}$$

Thus $f^*(r)$ is the maximum value of all numbers $f(x, r)$ as x runs through all x where $g_j(x, r) = 0$, $j = 1 \ldots m$.

The function $f^*(r)$ is called the value function for the problem. If we let $x_1^*(r), \ldots, x_n^*(r)$ denote the values of $x_1, \ldots, x_n$ for which the maximum value is obtained, then

$$f^*(r) = f(x^*(r), r) = f(x_1^*(r), \ldots x_n^*(r), r_1, \ldots, r_k)$$

When $n=1$, the construction of the function $f^*(r)$ as the "envelope" of all different $f(x,r)$ functions is indicated in ~~Fig 1~~ the following figure

The following result shows in general how to differentiate the value function.

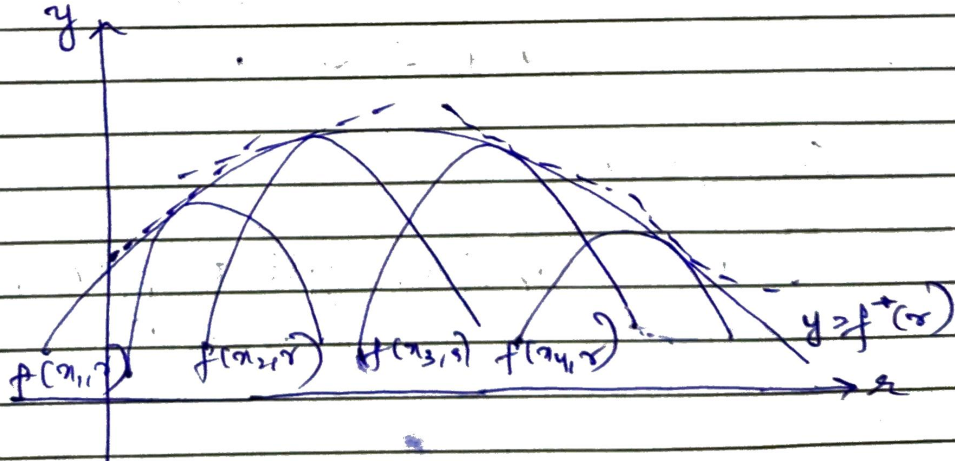
If $f^*(r) = \max_x f(x,r)$ and $x^*(r)$ is the value of $x$ that maximizes $f(x,r)$, then

$$\frac{\partial f^*(r)}{\partial r_j} = \frac{\partial f(x^*(r), r)}{\partial r_j} \qquad (j = 1 \dots k)$$



This is called envelope theorem.

Fig:- The function $f^*(r)$ is the envelope of all the different $f(x,r)$ functions



Non-linear programming

Simple case.

$$\max f(x,y) \text{ subject to } g(x,y) \leq c$$

Step 1: Associate a constant Lagrange multiplier $\lambda$ with the constraint $g(x,y) \leq c$ and define the Lagrangean function

$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c).$$

Step 2: Equate partials of $\mathcal{L}(x,y)$ to zero:

$$\mathcal{L}_1'(x,y) = f_1'(x,y) - \lambda g_1(x,y) = 0.$$
$$\mathcal{L}_2'(x,y) = f_2'(x,y) - \lambda g_2'(x,y) = 0$$



Step 3: Introduce the complementary slackness condition.

$$\lambda \geq 0 \; (= 0 \text{ if } g(x,y) < c)$$

Step 4: Require $(x,y)$ to satisfy the constraint

$$g(x,y) \leq c.$$

General case

$$\max(\min) f(x) \text{ subject to } g_j(x) \leq c_j \; (j = 1 \ldots m)$$

where $x$ denotes $(x_1, \ldots, x_n)$.

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1. Write down the Lagrangean

$$\mathcal{L}(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j)$$

with $\lambda_1, \ldots, \lambda_m$ as the Lagrange multipliers associated with $m$ constraints.

2. Equate all the first order partials of $\mathcal{L}(x)$ to 0:

$$\frac{\partial \mathcal{L}(x)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x)}{\partial x_i} = 0 \quad (i = 1, \ldots, n)$$

3. Impose the complimentary slackness conditions:

$$\lambda_j \geq 0 \; (= 0 \text{ if } g_j(x) < c_j) \quad (j = 1, \ldots, m)$$

4. Require $x$ to satisfy the constraints

$$g_j(x) \leq c_j \quad (j = 1, \ldots, m)$$

Find all $x$ together with associated values of $\lambda_1, \ldots, \lambda_m$ which satisfy all these conditions. These are the solution candidates, at least one of which solves the problem.

Steps 2 - 4 are often called the Kuhn Tucker conditions. These are the necessary (not sufficient) conditions for the solution of problem.

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More on Kuhn–Tucker conditions:

Provided that suitable concavity / convexity conditions are satisfied the Kuhn–Tucker conditions are sufficient for optimality

Consider the non-linear programming problem

$$\max f(x) \text{ subject to } g_j(x) \le c_j, \quad j = 1, \dots, m$$

where $f$ and $g_1 \dots g_m$ are continuously differentiable with $f$ concave and $g_1, \dots g_m$ all convex. Suppose that there exist numbers $\lambda_1, \dots \lambda_m$ and a feasible vector $x^0$ such that

(a) $$\frac{\partial f(x^0)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x^0)}{\partial x_i} \ge 0 \quad (i = 1, \dots n)$$

(b) $$\lambda_j \ge 0 \quad (= 0 \text{ if } g_j(x^0) < c_j) \quad (j = 1, \dots, m)$$

Then $x^0$ solves the problem.